STABILITY OF CIRCULAR SATELLITE ORBITS IN THE EARTH'S NORMAL GRAVITATIONAL FIELD WITH ALLOWANCE FOR THE DIPOLE MAGNETIC FIELD

PMM Vol. 31, No. 4, 1967, pp. 737-743

V.G. DEGTIAREV
(Leningrad)

(*Received February 28, 1967*)

Magnetic forces belong to the class of forces which have no noticeable effects on the character of satellite orbits. They are taken into account in problems on the motion of satellites near a center of mass, but are disregarded in problems on their orbital motion.

On the other hand, in stability studies one is obliged to take magnetic forces into account, since even very weak ones can give rise to unstable motion. In orbit correction problems, for example, it is also very important to know which forces have any marked effect on the orbits of artificial earth satellites and which forces are practically negligible.

Studies of the stability of circular motions in gravitational fields have been carried out by Chetaev [1], Duboshin [2], Demin [3 and 4], Aksenov, Grebenikov [4], and Degtiarev [5 to 7].

In the present paper we consider the stability of circular motions in the axisymmetrical gravitational and dipole magnetic fields of planets. The necessary and sufficient conditions for stability are derived. In the case of earth satellites it is proved that the equations of motion admit only of circular equatorial orbits and that the stability conditions are fulfilled for such orbits.

1. Formulation of the problem. The equations of motion. Let us consider the motion of a satellite of mass m in the gravitational and magnetic fields of a planet Our coordinate system is orthogonal and planetocentric; the z-axis is directed along the axis of rotation of the planet.

It is assumed that the gravitational axis has an axis of symmetry (the z-axis), i.e. that the potential of the gravitational field is of the form $U = U(\rho, z)$, where $\rho^2 = x^2 + y^2$.

An artificial satellite is likely to contain current-carrying conductors, batteries, capacitors, electromagnets, etc., all of which can subject it to magnetic forces. As before, in investigating the motion of the satellite center of mass we shall consider the satellite as a point of mass m equal to the total mass of the satellite; this point mass is assumed to carry the charge e equal to the total charge on the satellite. Then, by the Lorentz formula, the magnetic force is equal to $ec^{-1}[\mathbf{v} \times \mathbf{H}]$ where $\mathbf{v}$ is the satellite velocity vector, $\mathbf{H}$ is the magnetic field intensity vector, and c is the velocity of light. If the magnetic field has the potential V, then $\mathbf{H} = \operatorname{grad} V$. For the dipole magnetic field with z-axis we have $V = Mz/(\rho^2 + z^2)^{3/2}$, where M is the magnetic moment of the planet. In the case of the Earth, the dipole field constitutes the major portion of the magnetic field.

The vector differential equation of satellite motion in the gravitational and magnetic fields of a planet is of the form

$$\frac{d^2\mathbf{r}}{dt^2} = \operatorname{grad} U + \frac{e}{cm}[\mathbf{v} \times \mathbf{H}]$$

where $\mathbf{r}$ is the radius vector of the satellite. In the cylindrical coordinates ρ, ψ, z the equations of motion can be written as

$$\rho^{\cdot\cdot} - \rho\psi^{\cdot 2} = \frac{\partial U}{\partial \rho} + \frac{e}{cm}\rho\psi^{\cdot}\frac{\partial V}{\partial z}, \qquad z^{\cdot\cdot} = \frac{\partial U}{\partial z} - \frac{e}{cm}\rho\psi^{\cdot}\frac{\partial V}{\partial \rho}$$

$$(\rho^2\psi^{\cdot})^{\cdot} = \frac{e}{cm}\rho\left(\frac{\partial V}{\partial \rho}z^{\cdot} - \frac{\partial V}{\partial z}\rho^{\cdot}\right) \qquad \left(V = \frac{Mz}{(\rho^2+z^2)^{3/2}}\right)$$

Here and below a dot in the position of a prime indicates a derivative with respect to time. Now, replacing the angular velocity $\psi^{\cdot}$ by the doubled sectorial velocity $\sigma = \rho^2\psi^{\cdot}$ and taking account of the expression for V, we obtain

$$\rho^{\cdot\cdot} - \frac{\sigma^2}{\rho^3} = \frac{\partial U}{\partial \rho} + \frac{k\sigma(\rho^2 - 2z^2)}{\rho(\rho^2+z^2)^{5/2}}, \qquad z^{\cdot\cdot} = \frac{\partial U}{\partial z} + \frac{3k\sigma z}{(\rho^2+z^2)^{5/2}}$$
$$\sigma^{\cdot} = -\rho\left[\frac{3k\rho z}{(\rho^2+z^2)^{5/2}}z^{\cdot} + \frac{k(\rho^2-2z^2)}{(\rho^2+z^2)^{5/2}}\rho^{\cdot}\right] \qquad \left(k = \frac{eM}{cm}\right) \tag{1.1}$$

Hence, the circular orbits

$$\rho = \rho_0 = \text{const}, \; z = z_0 = \text{const}, \; \sigma = \sigma_0 = \rho_0^2\omega = \text{const} \tag{1.2}$$

are possible if the following conditions are fulfilled:

$$\frac{\sigma_0^2}{\rho_0^3} + \left(\frac{\partial U}{\partial \rho}\right)_0 + \frac{k\sigma_0(\rho_0^2 - 2z_0^2)}{\rho_0(\rho_0^2+z_0^2)^{5/2}} = 0, \qquad \left(\frac{\partial U}{\partial z}\right)_0 + \frac{3k\sigma_0 z_0}{(\rho_0^2+z_0^2)^{5/2}} = 0 \tag{1.3}$$

Circular orbits (1.2) are circles in the plane perpendicular to the axis of rotation of the planet with their centers on this axis.

The question of which circular orbits are possible in the Earth's gravitational and magnetic fields will be taken up in Section 4.

2. Necessary conditions for stability. Let us derive the stability conditions for circular orbits (1.2). We introduce the following notation for the perturbations:

$$\rho = \rho_0 + x_1, \qquad z = z_0 + x_2, \qquad \sigma = \sigma_0 + x_3, \qquad \rho^{\cdot} = x_1^{\cdot}, \qquad z^{\cdot} = x_2^{\cdot}$$

Substituting these values of $\rho^{\cdot}$, $\rho^{\cdot}$, z, $z^{\cdot}$, σ into Eqs. (1.1), we obtain the equations of perturbed motion,

$$x_1^{\cdot\cdot} - \frac{(\sigma_0+x_3)^2}{(\rho_0+x_1)^3} = \frac{\partial U(\rho_0+x_1 z_0+x_2)}{\partial(\rho_0+x_1)} + \frac{k(\sigma_0+x_3)[(\rho_0+x_1)^2 - 2(z_0+x_2)^2]}{(\rho_0+x_1)[(\rho_0+x_1)^2+(z_0+x_2)^2]^{5/2}}$$
$$x_2^{\cdot\cdot} = \frac{\partial U(\rho_0+x_1, z_0+x_2)}{\partial(z_0+x_2)} + \frac{3k(\sigma_0+x_3)(z_0+x_2)}{[(\rho_0+x_1)^2+(z_0+x_2)^2]^{5/2}} \tag{2.1}$$
$$x_3^{\cdot} = -(\rho_0+x_1)\left\{\frac{3k(\rho_0+x_1)(z_0+x_2)}{[(\rho_0+x_1)^2+(z_0+x_2)^2]^{5/2}}x_2^{\cdot} + \frac{k[(\rho_0+x_1)^2-2(z_0+x_2)^2]}{[(\rho_0+x_1)^2+(z_0+x_2)^2]^{5/2}}x_1^{\cdot}\right\}$$

Expanding the right-hand sides of Eqs. (2.1) in series in powers of x_1, x_2, x_3, $x_1^{\cdot}$, $x_2^{\cdot}$ and retaining first-order terms only, we obtain the equations for the first approximation of the perturbed motion,

$$x_1^{\cdot\cdot} - \alpha_{11}x_1 - \gamma_{12}x_2 - \xi_{13}x_3 = 0, \qquad -\gamma_{21}x_1 + x_2^{\cdot\cdot} - \beta_{22}x_2 - \xi_{12}x_3 = 0$$
$$\eta_{21}x_1^{\cdot} + \eta_{22}x_2^{\cdot} + x_3^{\cdot} = 0 \tag{2.2}$$

Here

$$\alpha_{11} = \left(\frac{\partial^2 U}{\partial \rho^2}\right)_0 + \frac{3}{\rho_0}\left(\frac{\partial U}{\partial \rho}\right) + \frac{3k\sigma_0(\rho_0^2 - 2z_0^2)}{\rho_0^2(\rho_0^2+z_0^2)^{5/2}} + \frac{k\sigma_0(-4\rho_0^4 + 13\rho_0^2 z_0^2 + 2z_0^4)}{\rho_0^2(\rho_0^2+z_0^2)^{7/2}}$$

$$\beta_{22} = \left(\frac{\partial^2 U}{\partial z^2}\right)_0 + \frac{3k\sigma_0(\rho_0^2 - 4z_0^2)}{(\rho_0^2+z_0^2)^{7/2}}, \qquad \gamma_{21} = \left(\frac{\partial^2 U}{\partial \rho \partial z}\right)_0 - \frac{15k\sigma_0\rho_0 z_0}{(\rho_0^2+z_0^2)^{7/2}} \tag{2.3}$$

$$\gamma_{12} = \left(\frac{\partial^2 U}{\partial \rho \partial z}\right)_0 + \frac{k\sigma_0(6z_0^2 - 9\rho_0^2)}{\rho_0(\rho_0^2+z_0^2)^{7/2}}, \qquad \xi_{13} = \frac{2\sigma_0}{\rho_0^3} + \frac{k(\rho_0^2 - 2z_0^2)}{\rho_0(\rho_0^2+z_0^2)^{5/2}}$$

$$\xi_{23} = \frac{3kz_0}{(\rho_0^2+z_0^2)^{5/2}}, \qquad \eta_{21} = \frac{k\rho_0(\rho_0^2 - 2z_0^2)}{(\rho_0^2+z_0^2)^{5/2}}, \qquad \eta_{22} = \frac{3k\rho_0^2 z_0}{(\rho_0^2+z_0^2)^{5/2}}$$

The characteristic equation of system (2.2),

$$\begin{vmatrix} \lambda^2-\alpha_{11} & -\gamma_{12} & -\xi_{13} \\ -\gamma_{21} & \lambda^2-\beta_{22} & -\xi_{23} \\ \eta_{21}\lambda & \eta_{22}\lambda & \lambda \end{vmatrix} = 0$$

can, with allowance for notation (2.3), be transformed into

$$\lambda\,[\lambda^4-(\alpha_0+\beta_0)\,\lambda^2+(\alpha_0\beta_0-\gamma_0^2)]=0 \tag{2.4}$$

$$\alpha_0=\alpha_{11}-\eta_{21}\xi_{13}=\left(\frac{\partial^2U}{\partial\rho^2}\right)_0+\frac{3}{\rho_0}\left(\frac{\partial U}{\partial\rho}\right)_0-\frac{3k\sigma_0(\rho_0^2-4z_0^2)}{(\rho_0^2+z_0^2)^{7/2}}-\frac{k^2(\rho_0^2-2z_0^2)^2}{(\rho_0^2+z_0^2)^5}$$

$$\beta_0=\beta_{22}-\eta_{22}\xi_{23}=\left(\frac{\partial^2U}{\partial z^2}\right)_0+\frac{3k\sigma_0(\rho_0^2-4z_0^2)}{(\rho_0^2+z_0^2)^{7/2}}-\frac{9k^2\rho_0^2z_0^2}{(\rho_0^2+z_0^2)^5} \tag{2.5}$$

$$\gamma_0=\gamma_{12}-\eta_{22}\xi_{13}=\gamma_{21}-\eta_{21}\xi_{23}=\left(\frac{\partial^2U}{\partial\rho\partial z}\right)_0-\frac{15k\sigma_0\rho_0z_0}{(\rho_0^2+z_0^2)^{7/2}}-\frac{3k^2\rho_0z_0(\rho_0^2-2z_0^2)}{(\rho_0^2+z_0^2)^5}$$

For the roots of Eq. (2.4) we have the expressions

$$\lambda_{1,2,3,4}=\pm\sqrt{{}^1\!/_2(\alpha_0+\beta_0)\pm\sqrt{{}^1\!/_4(\alpha_0+\beta_0)^2-(\alpha_0\beta_0-\gamma^2)}},\qquad \lambda_5=0$$

Let us prove that the necessary conditions for the stability of circular orbits (1.2) are

$$\alpha_0<0,\qquad \alpha_0\beta_0-\gamma_0^2>0 \tag{2.6}$$

We do this by proving that if at least one of inequalities (2.6) is violated, then solution (1.2) is unstable. If $\alpha_0\beta_0-\gamma_0^{\,2}<0$, then regardless of the sign of α_0 characteristic Eq. (2.4) has the positive root

$$\lambda=+\sqrt{{}^1\!/_2(\alpha_0+\beta_0)+\sqrt{{}^1\!/_4(\alpha_0+\beta_0)^2-(\alpha_0\beta_0-\gamma_0^2)}}$$

If, on the other hand, $\alpha_0>0$ but $\alpha_0\beta_0-\gamma_0^2>0$, then $\beta_0>0$ and there exists the positive root

$$\lambda=+\sqrt{{}^1\!/_2(\alpha_0+\beta_0)+\sqrt{{}^1\!/_4(\alpha_0-\beta_0)^2+\gamma^2}}$$

The case $\alpha_0=\beta_0=\gamma_0=0$ will not be considered. Now, by Liapunov's theorem on instability [8], if the characteristic equation has positive roots, then solution (1.2) is unstable on the basis of the first approximation.

3. Sufficient conditions for stability. These conditions will be obtained by considering the complete equations of perturbed motion (2.1). These equations admit of the existence of the first integrals, i.e. the energy integral F_1 and the area integral F_2,

$$F_1=\frac{1}{2}\left[x_1'^2+\frac{(\sigma_0+x_3)^2}{(\rho_0+x_1)^3}+x_2'^2\right]-U(\rho_0+x_1,z_0+x_2)-\frac{\sigma_0^2}{2\rho_0^3}+U(\rho_0,z^0)=\text{const}$$

$$F_2=x_3-\frac{k(\rho_0+x_1)^2}{[(\rho_0+x_1)^2+(z_0+x_2)^2]^{3/2}}+\frac{k\rho_0^2}{(\rho_0^2+z_0^2)^{3/2}}=\text{const} \tag{3.1}$$

Let us investigate stability by Liapunov's second method [8]. Following Chetaev [1], we construct the Liapunov function as a linearly quadratic sheaf of the first integrals (3.1),

$$W=F_1-\frac{\sigma_0}{\rho_0^2}F_2+AF_2^2$$

where A is a still undetermined quantity. Expanding W in a Maclaurin series in the variables x_1, x_1', x_2, x_2', x_3, making use of (1.3), and retaining second-order terms in x_1, x_1', x_2, x_2', x_3, we obtain

$$W={}^1\!/_2\,x_1'^2+{}^1\!/_2\,x_2'^2+{}^1\!/_2\,[-\alpha x_1^2-\beta x_2^2+(\rho_0^{-2}+2A)\,x_3^2-2\gamma x_1x_2-2\xi x_1x_3-2\eta_2x_2x_3]+\ldots \tag{3.2}$$

$$\alpha=\alpha_1-\alpha_2,\quad \alpha_1=\left(\frac{\partial^2U}{\partial\rho^2}\right)_0+\frac{3}{\rho_0}\left(\frac{\partial U}{\partial\rho}\right)_0+\frac{k\sigma_0(\rho_0^4+8\rho_0^2z_0^2-8z_0^4)}{\rho_0^2(\rho_0^2+z_0^2)^{7/2}}$$

$$\beta=\beta_1-\beta_2,\quad \beta_1=\left(\frac{\partial^2U}{\partial z^2}\right)_0+\frac{3k\sigma_0(\rho_0^2-4z_0^2)}{(\rho_0^2+z_0^2)^{7/2}}$$

$$\gamma=\gamma_1-\gamma_2,\quad \gamma_1=\left(\frac{\partial^2U}{\partial\rho\partial z}\right)_0-\frac{3k\sigma_0z_0(3\rho_0^2-2z_0^2)}{\rho_0(\rho_0^2+z_0^2)^{7/2}} \tag{3.3}$$

$$\xi = \xi_1 - \xi_2, \quad \xi_1 = \frac{2\sigma_0}{\rho_0^3}, \quad \alpha_2 = 2A\alpha_2^\circ, \quad \alpha_2^\circ = \frac{k^2\rho_0^2(\rho_0^2 - 2z_0^2)^2}{(\rho_0^2 + z_0^2)^5}$$

$$\beta_2 = 2A\beta_2^\circ, \quad \beta_2^\circ = \frac{9k^2\rho_0^4 z_0^2}{(\rho_0^2 + z_0^2)^5} \qquad \gamma_2 = 2A\gamma_2^\circ, \quad \gamma_2^\circ = \frac{3k^2\rho_0^3 z_0(\rho_0^2 - 2z_0^2)}{(\rho_0^2 + z_0^2)^5}$$

$$\xi_2 = 2A\xi_2^\circ, \quad \xi_2^\circ = \frac{k\rho_0(\rho_0^2 - 2z_0^2)}{(\rho_0^2 + z_0^2)^{5/2}} \qquad \eta_2 = 2A\eta_2^\circ, \quad \eta_2^\circ = -\frac{3k\rho_0^2 z_0}{(\rho_0^2 + z_0^2)^{5/2}}$$

But since $W^\cdot = 0$, the function W satisfies the conditions of Liapunov's stability theorem provided $W > 0$. For sufficiently small perturbations $x_1, x_2, x_1^\cdot, x_2^\cdot$, the sign of W is determined by the sign of the quadratic form appearing on the right-hand side of (3.2); the latter is in turn positively defined (by virtue of the Sylvester criterion) if the principal diagonal minors of the matrix

$$\left\| \begin{matrix} -\alpha & -\gamma & -\xi \\ -\gamma & -\beta & -\eta_2 \\ -\xi & -\eta_2 & 2A + \rho_0^{-2} \end{matrix} \right\|$$

are positive. This yields the three inequalities

$$\alpha < 0, \qquad \alpha\beta - \gamma^2 > 0 \tag{3.4}$$

$$(2A + \rho_0^{-2})(\alpha\beta - \gamma^2) + \alpha\eta_2^2 - 2\gamma\xi\eta_2 + \beta\xi^2 > 0 \tag{3.5}$$

It can be proved that (3.4) are fulfilled if the inequalities

$$\alpha_1 < 0, \qquad \alpha_1\beta_1 - \gamma_1^2 > 0 \tag{3.6}$$

are fulfilled.

In fact, if we take $A > 0$, then $\alpha_2 > 0$ and the second inequality of (3.4) follows from the first. The second inequality of (3.4) can be written as

$$(\alpha_1\beta_1 - \gamma_1^2) - (\alpha_1\beta_2 + \alpha_2\beta_1 - 2\gamma_1\gamma_2) > 0 \tag{3.7}$$

since $\alpha_2\beta_2 - \gamma_2^2 = 0$. Taking account of notation (3.3), we can rewrite the second term in (3.7) as

$$2A\frac{k^2\rho_0^2}{(\rho_0^2 + z_0^2)^5}[9\rho_0^2 z_0^2(-\alpha_1) - 2\cdot 3\rho_0 z_0(2z_0^2 - \rho_0^2)\gamma_1 + (2z_0^2 - \rho_0^2)^2(-\beta_1)]$$

It is clear from this that the latter expression is positive if (3.6) are fulfilled; hence, (3.6) imply (3.7) and therefore the second inequality of (3.4).

Now let us consider inequality (3.5). After a series of transformations (in the course of which the terms with A^2 and A^3 drop out) this inequality can be written as

$$2A\,[(\alpha_1\beta_1 - \gamma_1^2) - \rho_0^{-2}(\alpha_2^\circ\beta_1 + \alpha_1\beta_2^\circ - 2\gamma_1^\circ\gamma_2) - \beta_2^\circ\xi_1^2 - 2(\gamma_1\xi_1\eta_2^\circ + \beta_1\xi_1\xi_2^\circ)] +$$
$$+ \rho_0^{-2}(\alpha_1\beta_1 - \gamma_1^2) + \beta\xi_1^2 > 0$$

Clearly, we can choose an $A > 0$ so large that the latter inequality is fulfilled if the expression in square brackets is positive.

Thus, the conditions of positive definition of the function W (the sufficient conditions for stability) reduce to the three inequalities

$$\alpha_1 < 0, \ \alpha_1\beta_1 - \gamma_1^2 > 0$$

$$(\alpha_1\beta_1 - \gamma_1^2) - \rho_0^2(\alpha_2^\circ\beta_1 + \alpha_1\beta_2^\circ - 2\gamma_1\gamma_2^\circ) - 2(\gamma_1\xi_1\eta_2^\circ + \beta_1\xi_1\xi_2^\circ) - \beta_2^\circ\xi_1^2 > 0 \tag{3.8}$$

We note that when the sufficient conditions for stability are fulfilled, solution (1.2) is stable with respect to the quantities $\rho, \rho, z^\cdot, z^\cdot, \sigma, \psi^\cdot$. As regards the angular variable ψ, however, the circular orbits under consideration are unstable, since ψ is a cyclic variable. This means that a satellite in perturbed motion travels along an orbit which is nearly circular, although its angular distances ψ can differ substantially from those characteristic of unperturbed motion.

The above stability conditions are valid for the circular orbits of satellites moving around planets with axisymmetrical gravitational and dipole magnetic fields.

4. Circular orbits of artificial Earth satellites. The potential of the gravitational field of the terrestrial ellipsoid (i.e. of the Earth's normal gravitational field) is approximated quite closely by a certain specially chosen potential. We shall make use of an approximating potential of the form proposed by Aksenov, Grebenikov and Demin [9],

$$U = \frac{\mu}{2}\left[\frac{1}{\sqrt{\rho^2 + (z - id)^2}} + \frac{1}{\sqrt{\rho^2 + (z + id)^2}}\right] \qquad (i = \sqrt{-1})$$

Here μ is the gravitational constant ($\mu = 3.98602 \times 10^5\ km^3/sec^2$) [10] and d is the Earth's compression characteristic ($d = 210\ km$) [9].

Despite the fact that the Earth's magnetic axis does not coincide with its gravitational axis, the terrestrial magnetic field is approximated well by the field of a dipole. The Earth's magnetic moment M is equal to $8.3 \times 10^{25} CGS\mu$ [11].

Proceeding from the expression for the potential of the Earth's normal gravitational field and recalling the fact that $\sigma_0 = \rho_0{}^2\omega$, we can write Eqs. (1.3) for the circular motions of a satellite in the form

$$\begin{aligned} &\omega^2 - \frac{\mu}{2}\left\{\frac{1}{[\rho_0{}^2 + (z_0 - id)^2]^{3/2}} + \frac{1}{[\rho_0{}^2 + (z_0 + id)^2]^{3/2}}\right\} + \frac{k\omega(\rho_0{}^2 - 2z_0{}^2)}{(\rho_0{}^2 + z_0{}^2)^{5/2}} = 0 \\ &-\frac{\mu}{2}\left\{\frac{z_0 - id}{[\rho_0{}^2 + (z_0 - id)^2]^{3/2}} + \frac{z_0 + id}{[\rho_0{}^2 + (z_0 + id)^2]^{3/2}}\right\} + \frac{3k\omega\rho_0{}^2 z_0}{(\rho_0{}^2 + z_0{}^2)^{5/2}} = 0 \end{aligned} \tag{4.1}$$

It is clear that $\rho_0{}^2 + z_0{}^2 = r_0{}^2 > R^2$, where $R = 6378.16\ km$ [10]. What are the possible values of the "magnetic coefficient" $k = eM/cm$ for motion in the Earth's magnetic field? For any body the value of the specific charge e/m cannot exceed $e/m = 1.76 \times 10^7\ CGS\mu$ per gram of electron mass, since all electrization occurs through saturation by electrons or heavier ions. Hence, substituting into the expression for k the value of the Earth's magnetic moment M and the velocity of light $c = 3 \times 10^{10}\ cm/sec$, we find that $0 \leqslant k < k_{max}$, where $k_{max} = 4.8693 \times 10^{22}\ cm^3/sec = 4.6893 \times 10^7\ km^3/sec$.

Let us turn to the analysis of Eqs. (4.1). It is clear that for $z_0 = 0$ (equatorial orbits) the second of Eqs. (4.1) is fulfilled; fulfillment of the first of these equations can be guaranteed by the proper choice of ω and ρ_0. However, for $z_0 \neq 0$ in the real case ($r_0 > R$, $0 \leqslant < k < k_{max}$) Eqs. (4.1) are not fulfilled, i.e. other circular orbits are impossible.

Let us prove this. Reducing by z_0 in the second of Eqs. (4.1), we can obtain

$$\omega = \frac{\mu}{2}\frac{(\rho_0{}^2 + z_0{}^2)^{5/2}}{3k\rho_0{}^2}\left\{\frac{1 - z_0{}^{-1}id}{[\rho_0{}^2 + (z_0 - id)^2]^{3/2}} + \frac{1 + z_0{}^{-1}id}{[\rho_0{}^2 + (z_0 + id)^2]^{3/2}}\right\} \tag{4.2}$$

The first of Eqs. (4.1) together with the second one can be transformed into

$$\omega^2 - \frac{2k\omega}{(\rho_0{}^2 + z_0{}^2)^{3/2}} - \frac{\mu}{2}\frac{id}{z_0}\left\{\frac{1}{[\rho_0{}^2 + (z_0 - id)^2]^{3/2}} - \frac{1}{[\rho_0{}^2 + (z_0 + id^2)^{3/2}}\right\} = 0 \tag{4.3}$$

Introducing the complex notation

$$\frac{1}{[\rho_0{}^2 + (z_0 + id)^2]^{3/2}} = a + ib$$

we can write Eqs. (4.2) and (4.3) in the following form:

$$\omega = \frac{(\rho_0{}^2 + z_0{}^2)^{5/2}}{3k\rho_0{}^2}\left(\mu a + \mu b\frac{d}{z_0}\right) \qquad \left(a = \frac{a_1\sqrt{l + a_1} - b_1\sqrt{l - a_1}}{l^3\sqrt{2}}\right) \tag{4.4}$$

$$\omega^2 - \frac{2k\omega}{(\rho_0{}^2 + z_0{}^2)^{3/2}} + \mu b\frac{d}{z_0} = 0 \qquad \left(b = \frac{-b_1\sqrt{l + a_1} - a_1\sqrt{l - a_1}}{l^3\sqrt{2}}\right)$$

$$a_1 = \rho_0{}^2 + z_0{}^2 - d^2, \qquad b_1 = -2z_0 d, \qquad l = \sqrt{a_1{}^2 + b_1{}^2} \tag{4.5}$$

It is clear that

$$\mu b\ d\ /\ z_0 > 0 \tag{4.6}$$

This inequality is self-evident for $Z_0 < 0$, while for $z_0 > 0$ it follows from the inequality

$$1 - (b_1/a_1)^2 < \sqrt{1 + (b_1/a_1)^2}$$

Inequality (4.6) makes it possible to obtain from Eqs. (4.4) the inequality

$$\omega > \frac{(\rho_0{}^2 + z_0{}^2)^{5/2}}{3k\rho_0{}^2}\mu a, \quad \omega < \frac{2k}{(\rho_0{}^2 + z_0{}^2)^{3/2}} < \frac{2k_{max}}{R^3} \tag{4.7}$$

If $Z_0 > 0$, after some transformations and elimination of positive terms, we obtain the following expression from (4.7):

$$\omega > \frac{\mu}{3\sqrt{2}k}\left(1 - \frac{d^2}{\rho_0{}^2 + z_0{}^2}\right)\left(1 + 4\frac{d^2}{(\rho_0{}^2 + z_0{}^2)^{-5/4}}\right)^{-5/4} > \frac{\mu}{3\sqrt{2}K_{max}}\left(1 - \frac{d^2}{R^2}\right)\left(1 + 4\frac{d^2}{R^2}\right)^{-5/4} \tag{4.8}$$

For $Z_0 < 0$, we find from (4.7) that

$$\omega > \frac{\mu}{3\sqrt{2}\,k}\left(1 - \frac{d^2}{\rho_0^2 + z_0^2} - 2\frac{d}{(\rho_0^2 + z_0^2)^{1/2}}\right)\left(1 + 4\frac{d^2}{\rho_0^2 + z_0^2}\right)^{-5/4} >$$
$$> \frac{\mu}{3\sqrt{2}\,k}\left(1 - \frac{d^2}{R^2} - 2\frac{d}{R}\right)\left(1 + 4\frac{d^2}{R^2}\right)^{-5/4} \quad (4.9)$$

Substituting in numerical values of μ, d, R, K_{max}, we see that inequalities (4.8) and (4.9) contradict the second inequality of (4.7). Since all of our inequalities were derived from Eqs. (4.1) under the assumption that $z_0 \neq 0$, this contradiction indicates that circular orbits are impossible for $z_0 \neq 0$.

5. **Stability of circular equatorial orbits of artificial Earth satellites.** For $z_0 = 0$ the second of Eqs. (4.1) is fulfilled, while the first becomes

$$\omega^2 + \frac{k\omega}{\rho_0^3} - \frac{\mu}{(\rho_0^2 + d^2)^{3/2}} = 0 \quad (5.1)$$

This equation enables us to obtain two groups of values for ω,

$$\omega_1 = -\frac{k}{2\rho_0^3} + \left(\frac{k^2}{4\rho_0^6} + \frac{\mu}{(\rho_0^2 - d^2)^{3/2}}\right)^{1/2}, \qquad \omega_2 = -\frac{k}{2\rho_0^3} - \left(\frac{k^2}{4\rho_0^6} + \frac{\mu}{(\rho_0^2 - d^2)^{3/2}}\right)^{1/2}$$

Clearly, $\omega_1 > 0$, $\omega_2 < 0$, which corresponds to the motion of a satellite along a circular orbit in the direction of the Earth's rotation and in the opposite direction, respectively.

The necessary stability conditions (9) for circular equatorial orbits are of the form

$$\frac{\mu(\rho_0^2 - 4d^2)}{(\rho_0^2 - d^2)^{5/2}} + \frac{3k\omega}{\rho_0^3} + \frac{k^2}{\rho_0^6} > 0, \quad \frac{\mu(\rho_0^2 + d^2)}{(\rho_0^2 - d^2)^{5/2}} - \frac{3k\omega}{\rho_0^3} > 0 \quad (5.2)$$

These inequalities can be sufficient conditions if we add to them the inequality

$$\frac{\mu(\rho_0^2 - 4d^2)}{(\rho_0^2 - d^2)^{5/2}} - \frac{k\omega}{\rho_0^3} > 0 \quad (5.3)$$

Let us investigate the sufficient stability conditions. If $\omega = \omega_1 > 0$, then the first of inequalities (5.2) is fulfilled, while the second inequality and inequality (5.3) can be transformed into

$$\Phi_1 = \frac{\mu(\rho_0^2 + d^2)}{(\rho_0^2 - d^2)^{5/2}} + \frac{3k^2}{2\rho_0^6} - \frac{3k}{\rho_0^3}\left(\frac{k^2}{4\rho_0^6} + \frac{\mu}{(\rho_0^2 - d^2)^{3/2}}\right)^{1/2} > 0 \quad (5.4)$$

$$\Phi_2 = \frac{\mu(\rho_0^2 - 4d^2)}{(\rho_0^2 - d^2)^{5/2}} + \frac{k^2}{2\rho_0^6} - \frac{k}{\rho_0^3}\left(\frac{k^2}{4\rho_0^6} + \frac{\mu}{(\rho_0^2 - d^2)^{3/2}}\right)^{1/2} > 0 \quad (5.5)$$

If $\omega = \omega_2 < 0$, then the second of inequalities (5.2) and inequality (5.3) are fulfilled,

$$\Phi_3 = \frac{\mu(\rho_0^2 - 4d^2)}{(\rho_0^2 - d^2)^{5/2}} - \frac{k^2}{2\rho_0^6} - \frac{3k}{\rho_0^3}\left(\frac{k^2}{4\rho_0^2} + \frac{\mu}{(\rho_0^2 - d^2)^{3/2}}\right)^{1/2} > 0 \quad (5.6)$$

Thus, investigation of the stability of circular equatorial satellite orbits has been reduced to the analysis of inequalities (5.4) to (5.6). Let us estimate the quantities Φ_i,

$$\Phi_1 = \frac{\mu(\rho_0^2 + d^2)}{(\rho_0^2 - d^2)^{5/2}} + \frac{3k^2}{2\rho_0^6}\left[1 - \left(1 + \frac{4\mu\rho_0^6}{k^2(\rho_0^2 - d^2)^{3/2}}\right)^{1/2}\right] =$$
$$= \frac{\mu}{(\rho_0^2 - d^2)^{3/2}}\left[\frac{\rho_0^2 + d^2}{\rho_0^2 - d^2} - \frac{6}{1 + (1 + 4\mu\rho_0^6/k^2(\rho_0^2 - d^2)^{3/2})^{1/2}}\right] >$$
$$> \frac{\mu}{(\rho_0^2 - d^2)^{3/2}}\left[1 - \frac{6}{1 + (1 + 4\mu R^3/k_{max}^2)^{1/2}}\right]$$

$$\Phi_2 = \frac{\mu}{(\rho_0^2 - d^2)^{3/2}}\left[\frac{\rho_0^2 - 4d^2}{\rho_0^2 - d^2} - \frac{2}{1 + (1 + 4\mu\rho_0^6/k^2(\rho_0^2 - d^2)^{3/2})^{1/2}}\right] >$$
$$> \frac{\mu}{(\rho_0^2 - d^2)^{3/2}}\left[1 - \frac{3d^2}{R^2 - d^2} - \frac{2}{1 + (1 + 4\mu R^3/k_{max}^2)^{1/2}}\right]$$

$$\Phi_3 = \frac{\mu(\rho_0^2 - 4d^2)}{(\rho_0^2 - d^2)^{5/2}} + \frac{3k^2}{2\rho_0^6} - \frac{3k^2}{2\rho_0^6}\left(1 + \frac{4\mu\rho_0^6}{k^2(\rho_0^2 - d^2)^{3/2}}\right)^{1/2} - \frac{4k^2}{2\rho_0^6} >$$

$$> \frac{\mu}{(\rho_0^2 - d^2)^{3/2}} \left[1 - \frac{3d^2}{R^2 - d^2} - \frac{2}{1 + (1 + 4\mu R^3/k_{max}^2)^{1/2}} \right] - \frac{2k_{max}^2}{(\rho_0^2 - d^2)^{3/2} R^3}$$

On substituting μ, R, d, k_{max}^2 into the expressions in square brackets in the estimates for Φ_1 and Φ_2, we see that $\Phi_1 > 0$, $\Phi_2 > 0$. We can do the same for Φ_3, first bringing the positive quantity $(\rho_0{}^2 - d^2)^{3/2}$ outside the square brackets. Thus, the sufficient conditions for the stability of circular equatorial orbits are fulfilled.

The author is grateful to V.V. Rumiantsev for his critical comments.

BIBLIOGRAPHY

1. Chetaev, N.G., Stability of Motion. Papers on Analytical Mechanics. Izd. Akad. Nauk SSSR, Moscow, 1962 (pp. 21, 37)
2. Duboshin, G.N., On the stability of trivial solutions of several differential equations of celestial mechanics, II: On stability in a certain special case of nonsteady-state motion. Tr. Gos. Astronom. Inst. im. Shternberga Vol. 15, No. 1, 1945.
3. Demin, V.G., On the stability of circular orbits. Vestn. MGU, Ser. 3, Fiz.-Astr. No. 1, 1960.
4. Aksenov, E.P., Grebenikov, E.A. and Demin, V.G., Qualitative analysis of forms of motion in the problem of motion of an artificial Earth satellite in the Earth's normal gravitational field, in Artificial Earth Satellites, No. 16. Izd. Akad. Nauk SSSR, Moscow, 1963, (p. 173)
5. Degtiarev, V.G., On the stability of circular motions in the problem of two stationary centers. PMM Vol. 26, No. 4, 1962.
6. Degtiarev, V.G., On the stability of motion in the generalized problem of two stationary centers. PMM Vol. 26, No. 6, 1962.
7. Degtiarev, V.G., On the stability of the circular orbits of artificial Earth satellites. Inzh. zh. Vol. 3, No. 3, 1963.
8. Liapunov, A.M., The General Problem of the Stability of Motion. Gostekhizdat, Moscow-Leningrad, 1950,(pp. 82, 127)
9. Aksenov, E.P., Grebenikov, E.A. and Demin, V.G., General solution of the problem of the motion of an artificial satellite in the Earth's normal gravitational field, in Artificial Earth Satellites, No. 8, Izd. Akad. Nauk SSSR, 1961, (p. 64).
10. El'iasberg, P.E., Introduction to the Theory of Artificial Earth Satellite Motion. Izd. "Nauka", Moscow, 1965, (p. 315)
11. Ianovskii, B.M., Terrestrial Magnetism, 2nd ed. Gostekhizdat, Moscow, 1953, (p. 93).

Translated by A.Y.